ISRAEL JOURNAL OF MATHEMATICS 96 (1996), 281–297

CENTRAL POLYNOMIALS AND MATRIX INVARIANTS

BY

ANTONIO GIAMBRUNO AND ANGELA VALENTI*

Dipartimento di Matematica, Università di Palermo
Via Archirafi 34, 90123 Palermo, Italy
e-mail: giambruno@ipamat.math.unipa.it avalenti@ipamat.math.unipa.it

In memory of Shimshon Amitsur

ABSTRACT

Let K be a field, char $K = 0$ and $M_n(K)$ the algebra of $n \times n$ matrices over K. If $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\mu = (\mu_1, \ldots, \mu_m)$ are partitions of n^2 let

$$F^{\lambda,\mu} = \sum_{\sigma,\tau \in S_{n^2}} (\operatorname{sgn} \sigma\tau)\, x_{\sigma(1)} \cdots x_{\sigma(\lambda_1)} y_{\tau(1)} \cdots y_{\tau(\mu_1)} x_{\sigma(\lambda_1+1)} \cdots x_{\sigma(\lambda_1+\lambda_2)} y_{\tau(\mu_1+1)} \cdots y_{\tau(\mu_1+\mu_2)} \cdots x_{\sigma(\lambda_1+\cdots+\lambda_{m-1}+1)} \cdots x_{\sigma(n^2)} y_{\tau(\mu_1+\cdots+\mu_{m-1}+1)} \cdots y_{\tau(n^2)}$$

where $x_1, \ldots, x_{n^2}, y_1, \ldots, y_{n^2}$ are noncommuting indeterminates and S_{n^2} is the symmetric group of degree n^2.

The polynomials $F^{\lambda,\mu}$, when evaluated in $M_n(K)$, take central values and we study the problem of classifying those partitions λ, μ for which $F^{\lambda,\mu}$ is a central polynomial (not a polynomial identity) for $M_n(K)$.

We give a formula that allows us to evaluate $F^{\lambda,\mu}$ in $M_n(K)$ in general and we prove that if λ and μ are not both derived in a suitable way from the partition $\delta = (1, 3, \ldots, 2n-3, 2n-1)$, then $F^{\lambda,\mu}$ is a polynomial identity for $M_n(K)$. As an application, we exhibit a new class of central polynomials for $M_n(K)$.

* Research supported by a grant from MURST of Italy.
Received May 2, 1995 and in revised form July 31, 1995

1. Introduction

Let K be a field of characteristic zero and $K\{X\}$ the free algebra on the countable set $X = \{x_1, x_2, \ldots\}$. Let $M_n(K)$ be the algebra of $n \times n$ matrices over K.

Recall that an element $f(x_1, \ldots, x_n) \in K\{X\}$ is called a **central polynomial** for $M_n(K)$ if for all $A_1, \ldots, A_n \in M_n(K)$, $f(A_1, \ldots, A_n)$ lies in the center of $M_n(K)$ and f is not a polynomial identity for $M_n(K)$ (i.e., f takes on nonzero values).

The first central polynomials for $M_n(K)$ for any n were constructed by Formanek ([2]) and Razmyslov ([7]) with two different methods. Other central polynomials were produced by Halpin ([4]) by exploiting the methods of [7] and recently by Drenský ([1]) who constructed new central polynomials of minimal known degree for any n.

In this paper we study a class of polynomials associated to pairs of partitions of n^2 and, as a consequence, we construct a new class of central polynomials for $M_n(K)$ multilinear and alternating in two sets of variables.

Let $\{y_1, y_2, \ldots\}$ be a new set of noncommuting variables. Recall that if r is a positive integer, an (improper) **partition** of r is a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_m)$ such that $\sum_{i=1}^m \lambda_i = r$. We write $\lambda \models r$ and $h(\lambda) = m$. Let S_r be the symmetric group on $\{1, 2, \ldots, r\}$. For each pair of partitions $\lambda, \mu \models n^2$ such that $h(\lambda) = h(\mu) = m$ define the polynomial

$$
\begin{aligned}
F^{\lambda,\mu} = \sum_{\sigma,\tau \in S_{n^2}} (\operatorname{sgn} \sigma\tau)\, & x_{\sigma(1)} \cdots x_{\sigma(\lambda_1)} y_{\tau(1)} \cdots y_{\tau(\mu_1)} x_{\sigma(\lambda_1+1)} \\
& \cdots x_{\sigma(\lambda_1+\lambda_2)} y_{\tau(\mu_1+1)} \cdots y_{\tau(\mu_1+\mu_2)} \\
& \cdots x_{\sigma(\lambda_1+\cdots+\lambda_{m-1}+1)} \\
& \cdots x_{\sigma(n^2)} y_{\tau(\mu_1+\cdots+\mu_{m-1}+1)} \cdots y_{\tau(n^2)}.
\end{aligned}
$$

These polynomials were first introduced by Regev in [9]; in that paper the author studied the polynomial identities of the algebra $M_n(K)$ through its cocharacter sequence and the polynomials $F^{\lambda,\mu}$ arose naturally as polynomials associated to Young tableaux of rectangular frames of height n^2.

Since $F^{\lambda,\mu}$ is a multilinear polynomial which is alternating in the two sets of variables $\{x_1, \ldots, x_{n^2}\}$ and $\{y_1, \ldots, y_{n^2}\}$, it follows that for all $A_1, \ldots, A_{n^2}$, $B_1, \ldots, B_{n^2} \in M_n(K), F^{\lambda,\mu}(A_1, \ldots, A_{n^2}, B_1, \ldots, B_{n^2})$ lies in K, the center of $M_n(K)$ (see [9, Lemma 2.1]). This leads to the following

PROBLEM: *Classify all partitions $\lambda, \mu \models n^2$ for which $F^{\lambda,\mu}(x,y)$ is a central polynomial for $M_n(K)$.*

We will translate this problem into a problem of matrix invariants through the following easy observation: let tr denote the usual trace; since $F^{\lambda,\mu}$ takes only central values in $M_n(K)$, then $F^{\lambda,\mu}$ is a central polynomial for $M_n(K)$ if and only if $\mathrm{tr}(F^{\lambda,\mu})$ does not vanish in $M_n(K)$.

About previous results, Regev in [9] conjectured that if δ is the partition $(1,3,\ldots,2n-3,2n-1)$ then $F^{\delta,\delta}(x,y)$ is a central polynomial for $M_n(K)$; later this conjecture was verified by Formanek in [3]. We shall see that Regev's polynomial $F^{\delta,\delta}$ plays a fundamental role in the classification of the central polynomials of the type $F^{\lambda,\mu}$.

If $\lambda \models n^2$ we say that λ is **δ-derived** if, after a rearrangement of the parts of λ, either $\lambda = \delta$ or λ is obtained from δ by splitting some parts of δ into two or more parts.

We shall prove that if $\lambda, \mu \models n^2$ are such that $h(\lambda) = h(\mu)$ and either λ or μ is not δ-derived, then $F^{\lambda,\mu}$ is a polynomial identity for $M_n(K)$. Moreover, in case λ and μ are both δ-derived we shall give an explicit formula (involving characters of the symmetric group) through which it is possible to check whether $F^{\lambda,\mu}$ is a central polynomial or a polynomial identity. As an application we shall give a class of central polynomials corresponding to certain partitions of n^2 in n+1 parts.

Our proof will be based on Formanek's construction of certain homomorphisms of the ring of matrix invariants (see [3]) and we shall follow his approach closely.

2. The ring of invariants

Let W be the direct sum of r copies of $M_n(K)$. Let the group $\mathrm{GL}(n,K)$ act on W via (adjoint action)

$$(A_1,\ldots,A_r) \to (PA_1P^{-1},\ldots,PA_rP^{-1}),$$

where $A_1,\ldots,A_r \in M_n(K)$ and $P \in \mathrm{GL}(n,K)$. If $K[W]$ is the symmetric algebra of W over K, the ring of invariants $K[W]^{\mathrm{GL}(n,K)}$ is called the ring of invariants of r $n\times n$ matrices and is denoted $C(n,r)$. If W is replaced by an infinite number of copies of $M_n(K)$, then the fixed ring is called **the ring of invariants of $n \times n$ matrices** and is denoted by C.

The ring of invariants can be defined in terms of generic $n \times n$ matrices. Let $\{u_{ij}^{(l)} \mid 1 \le i, j \le n, l \ge 1\}$ be a set of independent commuting indeterminates over K and $K[u_{ij}^{(l)}]$ the polynomial algebra over K; for $l \ge 1$, $X_l = (u_{ij}^{(l)}) \in M_n(K[u_{ij}^{(l)}])$ is called a generic $n \times n$ matrix over K.

If $P \in \mathrm{GL}(n,K)$ and $PX_lP^{-1} = (\overline{u}_{ij}^{(l)})$, then the action of $\mathrm{GL}(n,K)$ on $K[u_{ij}^{(l)}]$ is given by $u_{ij}^{(l)} \to \overline{u}_{ij}^{(l)}$ and $C = K[u_{ij}^{(l)}]^{\mathrm{GL}(n,K)}$ is the ring of matrix invariants.

In invariant theory, a theorem giving a generating set for a ring of invariants is called a first fundamental theorem. In this case we have

THEOREM (First Fundamental Theorem of Matrix Invariants [6, Theorem 1.3]): *C is generated as a K-algebra by the traces $\mathrm{tr}(X_{i_1} \cdots X_{i_s})$ where $X_{i_1} \cdots X_{i_s}$ is a monomial in the generic matrices $X_1, X_2, \ldots$.*

A second fundamental theorem in invariant theory gives the relations among the invariants. Since char $K = 0$, the multilinear relations determine all relations and we will state a second fundamental theorem of matrix invariants giving multilinear relations among the generators. To state this theorem in a precise way we introduce some terminology.

Recall that $\lambda = (\lambda_1, \ldots, \lambda_m) \models r$ is a proper partition of r if $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m > 0$ and we write $\lambda \vdash r$. If KS_r is the group algebra of the symmetric group S_r over K then

$$KS_r = \bigoplus_{\lambda \vdash r} I_\lambda$$

where I_λ is the minimal two-sided ideal of KS_r corresponding to $\lambda \vdash r$ and we set

$$J(n,r) = \bigoplus_{\substack{\lambda \vdash r \\ h(\lambda) > n}} I_\lambda.$$

Let $\pi \in S_r$ and write π as a product of disjoint cycles (including 1-cycles)

$$\pi = (i_1 \cdots i_{k_1})(j_1 \cdots j_{k_2}) \cdots (l_1 \cdots l_{k_t}).$$

We define the trace monomials $T_\pi = T_\pi(X_1, \ldots, X_r) \in C$ associated to the permutation π as

$$T_\pi(X_1, \ldots, X_r) = \mathrm{tr}(X_{i_1} \cdots X_{i_{k_1}})\mathrm{tr}(X_{j_1} \cdots X_{j_{k_2}}) \cdots \mathrm{tr}(X_{l_1} \cdots X_{l_{k_t}}).$$

Let $C(r)$ be the subspace of C consisting of all the elements multilinear in $X_1, \ldots, X_r$.

THEOREM (Second Fundamental Theorem of Matrix Invariants [6, Theorem 4.3], [8, Proposition 1]): *If φ: $KS_r \to C(r)$ is the K-linear map defined by*

$$\sum_{\pi\in S_r} \alpha_\pi \pi \to \sum_{\pi\in S_r} \alpha_\pi T_\pi(X_1,\ldots,X_r),$$

then $\operatorname{Ker}\varphi = J(n,r)$.

Let us write $\overline{KS_r} = KS_r/J(n,r)$ and let Θ_r: $\overline{KS_r} \to C(r)$ be the corresponding linear isomorphism induced by φ. Then $C(r)$ becomes a left and right S_r-module (see [3]).

If we rename the first $2n^2$ generic matrices as $X_1,\ldots,X_{n^2},Y_1,\ldots,Y_{n^2}$, then our original problem can be translated into the following:

PROBLEM: *Classify all partitions $\lambda,\mu \models n^2$ such that*

$$\operatorname{tr}(F^{\lambda,\mu}(X_1,\ldots,X_{n^2},Y_1,\ldots,Y_{n^2}))$$

is a nonzero matrix invariant.

3. The discriminant

Recall that if $A_1 = (a_{ij}^{(1)}),\ldots,A_{n^2} = (a_{ij}^{(n^2)})$ are $n\times n$ matrices then the **discriminant** of $A_1,\ldots,A_{n^2}$ is the determinant of the $n^2\times n^2$ matrix whose i-th row is

$$(a_{11}^{(i)},a_{12}^{(i)},\ldots,a_{1n}^{(i)},a_{21}^{(i)},\ldots,a_{nn}^{(i)})$$

and it is denoted $\Delta(A) = \Delta(A_1,\ldots,A_{n^2})$.

If $X_1,\ldots,X_{n^2}$ are generic $n\times n$ matrices, then $\Delta(X_1,\ldots,X_{n^2})$ is a multilinear alternating function of the X_i's and $\Delta(X_1,\ldots,X_{n^2}) \in C(n^2)$ (see [3, Lemma 3]).

Now, if $\lambda = (\lambda_1,\ldots,\lambda_m) \models n^2$ let $\bar\lambda = (\lambda_m,\lambda_{m-1},\ldots,\lambda_1) \models n^2$ and, in general, if $\tau\in S_r$ set $\lambda_\tau = (\lambda_{\tau(1)},\ldots,\lambda_{\tau(m)})$. For $\lambda \models n^2$ define $A_\lambda \in C(n^2)$ by

$$A_\lambda(X_1,\ldots,X_{n^2}) = \sum_{\pi\in S_{n^2}} (\operatorname{sgn}\pi)\operatorname{tr}(X_{\pi(1)}\cdots X_{\pi(\lambda_1)})\operatorname{tr}(X_{\pi(\lambda_1+1)}\cdots X_{\pi(\lambda_1+\lambda_2)}) \cdots \operatorname{tr}(X_{\pi(\lambda_1+\cdots+\lambda_{m-1}+1)}\cdots X_{\pi(n^2)}).$$

Notice that our definition of A_λ for $\lambda \vdash n^2$ differs from the one in [3] in the order in which the generic matrices appear: in the terminology of [3] A_λ would be called $A_{\bar\lambda}$.

In the sequel we shall always write $\delta = (1,3,\ldots,2n-3,2n-1) \models n^2$ and, so, $\bar\delta = (2n-1,2n-3,\ldots,3,1) \vdash n^2$.

Remark: Let $\sigma = \begin{pmatrix} 1 & 2 & \cdots & r \\ i_1 & i_2 & \cdots & i_r \end{pmatrix} \in S_r$. Consider σ as the word $i_1 i_2 \cdots i_r$ and write $\sigma = B_1 B_2 \cdots B_t$ where each B_j is a subword consisting of increasing consecutive integers. Notice that if σ' is obtained from σ by exchanging two consecutive subwords one of even length then $\mathrm{sgn}\sigma = \mathrm{sgn}\sigma'$; while if both subwords have odd length then $\mathrm{sgn}\sigma = -\mathrm{sgn}\sigma'$. It follows that $\mathrm{sgn}\sigma = \mathrm{sgn}\tau$ where $\tau = B_{l_1} \cdots B_{l_s} B_{l_{s+1}} \cdots B_{l_t}$ and $B_{l_1}, \ldots, B_{l_s}$ are all subwords of σ among the B_j's of odd length ($l_1 < l_2 < \cdots < l_s$). Order now the words B_{l_j} ($1 \le j \le s$) in the obvious way, that is, by requiring that $B_i < B_j$ iff $a < b$ for all $a \in B_i$ and $b \in B_j$ and let $B_{k_1} < B_{k_2} < \cdots < B_{k_s}$. Then it is clear that

$$\mathrm{sgn}\sigma = \mathrm{sgn}\tau = \mathrm{sgn}\begin{pmatrix} k_1 & k_2 & \cdots & k_s \\ l_1 & l_2 & \cdots & l_s \end{pmatrix}.$$

The following result is a consequence of [3, Theorem 4]:

PROPOSITION 1: *Let $\lambda \models n^2, h(\lambda) = m$.*

(1) *If for all $\tau \in S_m, \lambda_\tau \neq \delta$, then $A_\lambda(X_1, \ldots, X_{n^2}) = 0$.*

(2) *If there exists $\tau \in S_m = S_n$ such that $\lambda_\tau = \delta$, then*

$$A_\lambda(X_1, \ldots, X_{n^2}) = (\mathrm{sgn}\tau) C_n \Delta(X_1, \ldots, X_{n^2}),$$

where C_n is the nonzero constant computed in [3].

Proof: Let $\lambda = (\lambda_1, \ldots, \lambda_m)$. If we set $\lambda_0 = 0$, we can write

$$\begin{aligned} A_\lambda(X_1, \ldots, X_{n^2}) &= \sum_{\pi \in S_{n^2}} (\mathrm{sgn}\pi) \mathrm{tr}(X_{\pi(1)} \cdots X_{\pi(\lambda_1)}) \mathrm{tr}(X_{\pi(\lambda_1+1)} \cdots X_{\pi(\lambda_1+\lambda_2)}) \\ &\qquad \cdots \mathrm{tr}(X_{\pi(\lambda_1+\cdots+\lambda_{m-1}+1)} \cdots X_{\pi(n^2)}) \\ &= \sum_{\pi \in S_{n^2}} (\mathrm{sgn}\pi) \mathrm{tr}(X_{\pi(\lambda_1+\cdots+\lambda_{\tau(1)-1}+1)} \cdots X_{\pi(\lambda_1+\cdots+\lambda_{\tau(1)})}) \mathrm{tr}(X_{\pi(\lambda_1+\cdots+\lambda_{\tau(2)-1}+1)} \\ &\qquad \cdots X_{\pi(\lambda_1+\cdots+\lambda_{\tau(2)})}) \cdots \mathrm{tr}(X_{\pi(\lambda_1+\cdots+\lambda_{\tau(m)-1}+1)} \cdots X_{\pi(\lambda_1+\cdots+\lambda_{\tau(m)})}) \\ &= (\mathrm{sgn}\tau') \sum_{\pi \in S_{n^2}} (\mathrm{sgn}\pi) \mathrm{tr}(X_{\pi(1)} \cdots X_{\pi(\lambda_{\tau(1)})}) \cdots \mathrm{tr}(X_{\pi(\lambda_{\tau(1)}+1)} \cdots X_{\pi(\lambda_{\tau(1)}+\lambda_{\tau(2)})}) \\ &\qquad \cdots \mathrm{tr}(X_{\pi(\lambda_{\tau(1)}+\cdots+\lambda_{\tau(m)-1}+1)} \cdots X_{\pi(n^2)}) \\ &= (\mathrm{sgn}\tau') A_{\lambda_\tau}(X_1, \ldots, X_{n^2}), \end{aligned}$$

where τ' is a permutation computed according to the previous remark. We now apply [3, Theorem 4]: if $\lambda_\tau \neq \delta$, for all $\tau \in S_m$, then $A_\lambda(X) = 0$; if there exists

$\tau \in S_m$ such that $\lambda_\tau = \delta$ then, since all the parts of δ are of odd length, $(\mathrm{sgn}\tau')$ $= (\mathrm{sgn}\tau)$ and we have

$$A_\lambda(X) = (\mathrm{sgn}\tau)A_{\lambda_\tau}(X) = (\mathrm{sgn}\tau)A_\delta(X) = (\mathrm{sgn}\tau)C_n\Delta(X). \qquad \blacksquare$$

4. δ-derived partitions

We fix some notation: if $\rho \in S_m$ we write ρ as a product of disjoint cycles, including 1-cycles,

$$\rho = (i_1 \cdots i_{s_1})(j_1 \cdots j_{s_2}) \cdots (l_1 \cdots l_{s_r})$$

and we make this notation unique by requiring that $i_1, j_1, \dots, l_1$ are the least elements in each cycle and $i_1 < j_1 < \cdots < l_1$.

Let now $\lambda \models n^2, h(\lambda) = m$ and let $\rho \in S_m$ be written as above. We define a new partition $\lambda^{(\rho)} \models n^2$ by setting

$$\lambda^{(\rho)} = (\lambda_{i_1} + \cdots + \lambda_{i_{s_1}}, \lambda_{j_1} + \cdots + \lambda_{j_{s_2}}, \dots, \lambda_{l_1} + \cdots + \lambda_{l_{s_r}});$$

we also write $\lambda^{(\rho)} = (\lambda_1^{(\rho)}, \dots, \lambda_r^{(\rho)})$.

We say that a partition $\lambda \models n^2, h(\lambda) = m$ is **δ-derived** if there exist permutations $\rho \in S_m$ and $\tau \in S_n$ such that $(\lambda^{(\rho)})_\tau = \delta$. In other words λ is δ-derived if there exists $\tau \in S_m$ such that either $\lambda_\tau = \delta$ or λ_τ is obtained from δ by splitting some parts of δ into two or more parts. For a δ-derived partition $\lambda \models n^2, h(\lambda) = m$, we define

$$B_\lambda = \{\rho \in S_m \mid (\lambda^{(\rho)})_\tau = \delta, \text{ for some } \tau \in S_n\}.$$

Let $\lambda \models n^2$ be δ-derived, $\rho \in B_\lambda$ and $(\lambda^{(\rho)})_\tau = \delta$. Write ρ as a product of disjoint cycles

$$\rho = (i_1 \dots i_{s_1})(j_1 \dots j_{s_2}) \cdots (l_1 \dots l_{s_r}),$$

and let $\sigma \in S_{n^2}$ be the permutation defined by the rule

$$\sigma(a) = \begin{cases} \lambda_1 + \lambda_2 + \cdots + \lambda_{i_t - 1} + a & \text{if } \lambda_{i_1} + \cdots + \lambda_{i_{t-1}} + 1 \le a \le \lambda_{i_1} + \cdots + \lambda_{i_t} \\ & \text{for } t = 1, \dots, s_1 \\ \lambda_1 + \lambda_2 + \cdots + \lambda_{j_t - 1} + a & \text{if } \lambda_{i_1} + \cdots + \lambda_{i_{s_1}} + \lambda_{j_1} + \cdots + \lambda_{j_{t-1}} + 1 \\ & \quad \le a \le \lambda_{i_1} + \cdots + \lambda_{i_{s_1}} + \lambda_{j_1} + \cdots + \lambda_{j_t} \\ & \text{for } t = 1, \dots, s_2 \\ \vdots & \\ \lambda_1 + \lambda_2 + \cdots + \lambda_{l_t - 1} + a & \text{if } \lambda_{i_1} + \cdots + \lambda_{j_1} + \cdots + \lambda_{l_1} + \cdots \lambda_{l_{t-1}} + 1 \\ & \quad \le a \le n^2 \\ & \text{for } t = 1, \dots, s_r \end{cases}$$

where we are assuming $\lambda_0 = \lambda_{j_0} = \cdots = \lambda_{l_0} = 0$. We then define $\varepsilon(\lambda, \rho) = (\mathrm{sgn}\sigma)(\mathrm{sgn}\tau)$. We remark that the sign of σ is computed according to the remark above. With the notation introduced above in mind we prove the following

LEMMA 1: *Let $\lambda \models n^2, h(\lambda) = m$ and $\rho \in S_m$. Then*

$$\sum_{\pi\in S_{n^2}} (\mathrm{sgn}\pi)T_\rho(X_{\pi(1)}\cdots X_{\pi(\lambda_1)}, X_{\pi(\lambda_1+1)}\cdots X_{\pi(\lambda_1+\lambda_2)}, \dots, X_{\pi(\lambda_1+\dots+\lambda_{m-1}+1)}\cdots X_{\pi(n^2)}) = \begin{cases} \epsilon(\lambda,\rho)C_n\Delta(X_1,\dots,X_{n^2}) & \text{if } \lambda \text{ is } \delta\text{-derived and } \rho \in B_\lambda \\ 0 & \text{otherwise.}\end{cases}$$

Proof: Write $\rho = (i_1\cdots i_{s_1})(j_1\cdots j_{s_2})\cdots(l_1\cdots l_{s_r})$ as a product of disjoint cycles. We have:

$$\begin{aligned}
&\sum_{\pi\in S_{n^2}} (\mathrm{sgn}\pi)T_\rho(X_{\pi(1)}\cdots X_{\pi(\lambda_1)}, X_{\pi(\lambda_1+1)}\cdots X_{\pi(\lambda_1+\lambda_2)}, \dots, X_{\pi(\lambda_1+\cdots+\lambda_{m-1}+1)}\cdots X_{\pi(n^2)})\\
&= \sum_{\pi\in S_{n^2}} (\mathrm{sgn}\pi)\mathrm{tr}(X_{\pi(\lambda_1+\cdots+\lambda_{i_1-1}+1)}\cdots X_{\pi(\lambda_1+\cdots+\lambda_{i_1})}\\
&\quad\cdots X_{\pi(\lambda_1+\cdots+\lambda_{i_{s_1}-1}+1)}\cdots X_{\pi(\lambda_1+\cdots+\lambda_{i_{s_1}})})\cdots\mathrm{tr}(X_{\pi(\lambda_1+\cdots+\lambda_{l_1-1}+1)}\\
&\quad\cdots X_{\pi(\lambda_1+\cdots+\lambda_{l_1})}\cdots X_{\pi(\lambda_1+\cdots+\lambda_{l_{s_r}-1}+1)}\cdots X_{\pi(\lambda_1+\cdots+\lambda_{l_{s_r}})})\\
&= \sum_{\pi\in S_{n^2}} (\mathrm{sgn}\pi)\mathrm{tr}(X_{\pi\sigma^{-1}(1)}\cdots X_{\pi\sigma^{-1}(\lambda_{i_1})}\cdots X_{\pi\sigma^{-1}(\lambda_{i_1}+\cdots+\lambda_{i_{s_1}-1}+1)}\\
&\quad\cdots X_{\pi\sigma^{-1}(\lambda_{i_1}+\cdots+\lambda_{i_{s_1}})})\cdots\mathrm{tr}(X_{\pi\sigma^{-1}(\lambda_{i_1}+\cdots+\lambda_{j_1}+\cdots+\lambda_{l_1-1}+1)}\cdots X_{\pi\sigma^{-1}(n^2)})\\
&= A_{\lambda^{(\rho)}}(X_{\sigma^{-1}(1)},\dots,X_{\sigma^{-1}(n^2)})\\
&= (\mathrm{sgn}\sigma)A_{\lambda^{(\rho)}}(X_1,\dots,X_{n^2})
\end{aligned}$$

where σ is the permutation defined right before the Lemma. By Proposition 1, $A_{\lambda^{(\rho)}} = 0$ unless $(\lambda^{(\rho)})_\tau = \delta$ for some $\tau \in S_n$, i.e., unless λ is δ-derived and $\rho \in B_\lambda$. The conclusion now follows from Proposition 1. ∎

5. The function Φ^λ: $C(m) \to C(m)$

We now rename the first $m + n^2$ generic matrices $X_1, \dots, X_m, Y_1, \dots, Y_{n^2}$, and we let S_m be the symmetric group on $\{1, \dots, m\}$, S_{n^2} the symmetric group on $\{1^*, \dots, n^{2*}\}$ and S_{m+n^2} the symmetric group on $\{1, \dots, m, 1^*, \dots, n^{2*}\}$.

If $\lambda \models n^2, h(\lambda) = m$, define the function

$$\Gamma^\lambda: C(m) \to C(m+n^2)$$

such that

$$\Gamma^\lambda(f) = \sum_{\pi \in S_{n^2}} (\text{sign}\pi) f(X_1 Y_{\pi(1^*)} \cdots Y_{\pi(\lambda_1^*)}, X_2 Y_{\pi(\lambda_1^*+1^*)} \cdots Y_{\pi(\lambda_1^*+\lambda_2^*)}, \ldots, X_m Y_{\pi(\lambda_1^*+\cdots+\lambda_{m-1}^*+1^*)} \cdots Y_{\pi(n^{2*})}).$$

The following Lemma holds

Lemma 2 ([3, Lemma 6]): *If $f \in C(m)$, then there exists a unique $\hat{f} = \hat{f}(X_1, \ldots, X_m) \in C(m)$ such that $\Gamma^\lambda(f) = \hat{f}\Delta(Y_1, \ldots, Y_{n^2})$.*

Define maps

$$\Phi^\lambda: C(m) \to C(m) \quad \text{and} \quad \Phi_0^\lambda: \overline{KS_m} \to \overline{KS_m}$$

such that $\Phi^\lambda(f) = \hat{f}$ and $\Phi_0^\lambda = \Theta_m^{-1}\Phi^\lambda\Theta_m$. We have:

Proposition 2 ([3, Theorem 8]): *The maps Φ^λ and Φ_0^λ are left S_m-module homomorphisms.*

As in [3] we now compute $\Phi^\lambda(T_1)$. Let us write

$$\Phi^\lambda(T_1) = \sum_{\sigma \in S_m} a_\sigma T_\sigma.$$

Since Φ^λ is a left S_m-module homomorphism, for all $\rho \in S_m$ we have

$$\Phi^\lambda(T_\rho(X_1, \ldots, X_m)) = \sum_{\sigma \in S_m} a_\sigma T_{\rho\sigma}(X_1, \ldots, X_m).$$

Hence, if I denotes the identity $n \times n$ matrix,

$$\Phi^\lambda(T_\rho)(I, \ldots, I) = \sum_{\sigma \in S_m} a_\sigma T_{\rho\sigma}(I, \ldots, I).$$

For $\tau \in S_m$, let $z(\tau)$ denote the number of cycles in the decomposition of τ into disjoint cycles. Then, since $T_\tau(I, \ldots, I) = m^{z(\tau)}$, we get

$$\Phi^\lambda(T_\rho)(I, \ldots, I) = \sum_{\sigma \in S_m} a_\sigma m^{z(\rho\sigma)}.$$

We now prove

LEMMA 3: *Let $\lambda \models n^2, h(\lambda) = m$ and $\rho \in S_m$. Then*

$$\sum_{\sigma \in S_m} a_\sigma m^{z(\rho\sigma)} = \begin{cases} \epsilon(\lambda, \rho) C_n & \text{if } \lambda \text{ is } \delta\text{-derived and } \rho \in B_\lambda \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Write $\Delta(Y) = \Delta(Y_1, \dots, Y_{n^2})$. Recalling the definition of Γ^λ and Φ^λ we get

$$\begin{aligned} \Phi^\lambda(T_\rho)(I, \dots, I)\Delta(Y) =& \Gamma^\lambda(T_\rho)(I, \dots, I, Y_1, \dots, Y_{n^2}) \\ =& \sum_{\pi \in S_{n^2}} (\mathrm{sgn}\pi) T_\rho(Y_{\pi(1^*)} \cdots Y_{\pi(\lambda_1^*)}, Y_{\pi(\lambda_1^*+1^*)} \\ & \cdots Y_{\pi(\lambda_1^*+\lambda_2^*)}, \dots, Y_{\pi(\lambda_1^*+\dots+\lambda_{m-1}^*+1^*)} \cdots Y_{\pi(n^{2*})}). \end{aligned}$$

By Lemma 1 we obtain that

$$\Phi^\lambda(T_\rho)(I, \dots, I)\Delta(Y) = \begin{cases} \epsilon(\lambda, \rho) C_n \Delta(Y) & \text{if } \lambda \text{ is } \delta\text{-derived and } \rho \in B_\lambda \\ 0 & \text{otherwise.} \end{cases}$$

The proof now is completed by recalling that $\Phi^\lambda(T_\rho)(I, \dots, I) = \sum_{\sigma \in S_m} a_\sigma m^{z(\rho\sigma)}$. ∎

Since $\Phi^\lambda(T_1) = \sum_{\sigma \in S_m} a_\sigma T_\sigma$ then

$$\Phi_0^\lambda(1) = \overline{\sum_{\sigma \in S_m} a_\sigma \sigma} \in \overline{KS_m}.$$

We now apply the previous lemma; let us write

$$\begin{aligned} \left(\sum_{\sigma \in S_m} a_\sigma \sigma\right)\left(\sum_{\rho \in S_m} m^{z(\rho)}\rho\right) &= \sum_{\sigma, \rho \in S_m} a_\sigma m^{z(\rho)} \sigma \rho^{-1} \\ &= \sum_{\sigma, \rho \in S_m} a_\sigma m^{z(\rho\sigma)} \rho^{-1} \\ &= \sum_{\rho \in S_m} \left(\sum_{\sigma \in S_m} a_\sigma m^{z(\rho\sigma)}\right) \rho^{-1} \\ &= \begin{cases} C_n \sum_{\tau \in B_\lambda} \epsilon(\lambda, \tau)\tau^{-1} & \text{if } \lambda \text{ is } \delta\text{-derived} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since

$$\sum_{\rho \in S_m} m^{z(\rho)} \rho \in Z(KS_m),$$

where $Z(KS_m)$ is the center of KS_m, and $\sum_{\rho\in S_m} m^{z(\rho)}\rho$ is invertible (see [3, p. 103]), we get

$$(1)\quad \sum_{\sigma\in S_m} a_\sigma\sigma = \begin{cases} C_n(\sum_{\rho\in S_m} m^{z(\rho)}\rho)^{-1}\sum_{\tau\in B_\lambda}\epsilon(\lambda,\tau)\tau^{-1} & \text{if } \lambda \text{ is } \delta\text{-derived} \\ 0 & \text{otherwise.}\end{cases}$$

A formula for $(\sum_{\rho\in S_m} m^{z(\rho)}\rho)^{-1}$ can be derived as in [3] and we now describe it.

Let V be a K-vector space, $\dim_K V = m$ and let S_m act on the tensor space $V^{\otimes m}$ by place permutation, i.e.,

$$\sigma(v_1\otimes\cdots\otimes v_m) = v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(m)}.$$

Call P the character of the induced S_m-representation. Then if $\alpha\vdash m$ and χ_α is the corresponding irreducible S_m-character, by [5, p. 192] we get that the multiplicity of χ_α in P is given by the formula

$$\langle P,\chi_\alpha\rangle = \prod_{i,j}\left(\frac{m-i+j}{h^\alpha_{ij}}\right)$$

where (i,j) are the coordinates of the nodes of the diagram of α and h^α_{ij} are the associated hook numbers. Recall that $\langle P,\chi_\alpha\rangle$ is also the number of semistandard tableaux of shape α and content $1,2,\ldots,m$. By [3, p. 104] it follows that

$$\left(\sum_{\rho\in S_m} m^{z(\rho)}\rho\right)^{-1} = \frac{1}{(m!)^2}\sum_{\rho\in S_m}\sum_{\alpha\vdash m}\frac{\chi_\alpha(1)^2\chi_\alpha(\rho)}{\langle P,\chi_\alpha\rangle}\rho.$$

Thus recalling that $\Phi^\lambda(T_1) = \sum a_\sigma T_\sigma$, from (1) we get that $\Phi^\lambda(T_1) = 0$ unless λ is δ-derived and, in this last case,

$$\Phi^\lambda(T_1) = \frac{C_n}{(m!)^2}\sum_{\rho\in S_m}\sum_{\tau\in B_\lambda}\epsilon(\lambda,\tau)\sum_{\alpha\vdash m}\frac{\chi_\alpha(1)^2\chi_\alpha(\rho)}{\langle P,\chi_\alpha\rangle}T_{\rho\tau^{-1}}.$$

Hence we can write

Proposition 3: *Let $\lambda \models n^2, h(\lambda) = m$. If λ is not δ-derived, $\Phi^\lambda(T_1) = 0$. If λ is δ-derived then*

$$\Phi^\lambda(T_1) = \frac{C_n}{(m!)^2}\sum_{\sigma\in S_m}\left(\sum_{\tau\in B_\lambda}\sum_{\alpha\vdash m}\frac{\chi_\alpha(1)^2\chi_\alpha(\sigma\tau)}{\langle P,\chi_\alpha\rangle}\epsilon(\lambda,\tau)\right)T_\sigma.$$

6. The main theorem

We can now prove the main result.

THEOREM 1: *Let $\lambda, \mu \models n^2, h(\lambda) = h(\mu) = m$. If either λ or μ is not δ-derived then $F^{\lambda,\mu}(x,y)$ is a polynomial identity for $M_n(K)$. If both λ and μ are δ-derived then $F^{\lambda,\mu}(x,y)$ is a central polynomial for $M_n(K)$ if and only if*

$$\sum_{\substack{\sigma\in B_\lambda\\ \tau\in B_\mu}}\sum_{\alpha\vdash m}\frac{\chi_\alpha(1)^2\chi_\alpha((mm-1\cdots 21)\sigma\tau)}{\langle P,\chi_\alpha\rangle}\epsilon(\lambda,\sigma)\epsilon(\mu,\tau)\neq 0.$$

Proof: As we pointed out before $F^{\lambda,\mu}(x,y)$ is a central polynomial if and only if

$$\mathrm{tr}(F^{\lambda,\mu}(X_1,\ldots,X_n^2,Y_1,\ldots,Y_n^2))$$

is a non-zero invariant.

Write $\Phi^\mu(T_1)=\sum_{\sigma\in S_m}a_\sigma T_\sigma$; then, for all $\rho\in S_m$, $\Phi^\mu(T_\rho)=\sum_{\sigma\in S_m}a_\sigma T_{\rho\sigma}$ and we compute

$$\begin{aligned}\sum_{\pi\in S_{n^2}}&(\mathrm{sgn}\pi)\mathrm{tr}(X_1Y_{\pi(1)}\cdots Y_{\pi(\mu_1)}X_2Y_{\pi(\mu_1+1)}\cdots Y_{\pi(\mu_1+\mu_2)}\\ &\qquad\cdots X_mY_{\pi(\mu_1+\cdots+\mu_{m-1}+1)}\cdots Y_{\pi(n^2)})\\ &=\Phi^\mu(T_{(12\cdots m)})\Delta(Y)\\ &=\sum_{\sigma\in S_m}a_\sigma T_{(12\cdots m)\sigma}(X_1,\ldots,X_m)\Delta(Y).\end{aligned}$$

Now make the substitutions

$$\begin{aligned}X_1&\to X_1X_2\cdots X_{\lambda_1}\\ X_2&\to X_{\lambda_1+1}\cdots X_{\lambda_1+\lambda_2}\\ &\vdots\\ X_m&\to X_{\lambda_1+\cdots+\lambda_{m-1}+1}\cdots X_n^2\end{aligned}$$

and skewsymmetrize with respect to $X_1,\ldots,X_n^2$. We get

$$\begin{aligned}\sum_{\tau,\pi\in S_{n^2}}&(\mathrm{sgn}\tau)(\mathrm{sgn}\pi)\mathrm{tr}(X_{\tau(1)}\cdots X_{\tau(\lambda_1)}Y_{\pi(1)}\cdots Y_{\pi(\mu_1)}X_{\tau(\lambda_1+1)}\cdots\\ &X_{\tau(\lambda_1+\lambda_2)}Y_{\pi(\mu_1+1)}\cdots Y_{\pi(\mu_1+\mu_2)}\cdots\\ &X_{\tau(\lambda_1+\cdots+\lambda_{m-1}+1)}\cdots X_{\tau(n^2)}Y_{\pi(\mu_1+\cdots+\mu_{m-1}+1)}\cdots Y_{\pi(n^2)})\\ =\sum_{\sigma\in S_m}a_\sigma&\sum_{\tau\in S_{n^2}}(\mathrm{sgn}\tau)T_{(12\cdots m)\sigma}(X_{\tau(1)}\cdots X_{\tau(\lambda_1)},X_{\tau(\lambda_1+1)}\\ &\cdots X_{\tau(\lambda_1+\lambda_2)},\ldots,X_{\tau(\lambda_1+\cdots+\lambda_{m-1}+1)}\cdots X_{\tau(n^2)})\Delta(Y).\end{aligned}$$

Observe that in the last equality the left hand side is equal to $\mathrm{tr}(F^{\lambda,\mu}(X,Y)) = \mathrm{tr}(F^{\lambda,\mu}(X_1,\ldots,X_{n^2},Y_1,\ldots,Y_{n^2}))$.

If μ is not δ-derived, then by the previous proposition $\Phi^{\mu}(T_1)=0$ and by the above computation also $\mathrm{tr}(F^{\lambda,\mu}(X,Y))=0$; hence $F^{\lambda,\mu}(x,y)$ is a polynomial identity in this case. If λ is not δ-derived then by Lemma 1

$$\sum_{\tau\in S_{n^2}}(\mathrm{sgn}\tau)T_{(12\cdots m)\sigma}(X_{\tau(1)}\cdots X_{\tau(\lambda_1)},\ldots,X_{\tau(\lambda_1+\cdots+\lambda_{m-1}+1)}\cdots X_{\tau(n^2)})=0$$

and also in this case $\mathrm{tr}(F^{\lambda,\mu}(X,Y))=0$ and $F^{\lambda,\mu}(x,y)$ is a polynomial identity for $M_n(K)$.

Suppose then that λ and μ are both δ-derived. Then by Lemma 1 and Proposition 3 we get

$$\begin{aligned}&\mathrm{tr}(F^{\lambda,\mu}(X,Y))\\&=C_n\sum_{(12\cdots m)\sigma\in B_\lambda}a_\sigma\epsilon(\lambda,(12\cdots m)\sigma)\Delta(X)\Delta(Y)\\&=\frac{C_n^2}{(m!)^2}\sum_{\substack{(12\cdots m)\sigma\in B_\lambda\\ \tau\in B_\mu}}\sum_{\alpha\vdash m}\frac{\chi_\alpha(1)^2\chi_\alpha(\sigma\tau)}{\langle P,\chi_\alpha\rangle}\epsilon(\lambda,(12\cdots m)\sigma)\epsilon(\mu,\tau)\Delta(X)\Delta(Y)\end{aligned}$$

and the conclusion of the theorem follows. ∎

7. Consequences

We now consider a special case of the previous theorem.

COROLLARY 1: *Let $\lambda\models n^2$ be obtained from δ by splitting the k-th part (of length $2k-1$) into two parts of length, say, 2(k-t) and $2t-1$ respectively. Then $F^{\lambda,\lambda}(x,y)$ is a central polynomial provided*

$$\sum_{\alpha\vdash n+1}\frac{\chi_\alpha(1)^2(\chi_\alpha(\sigma)-\chi_\alpha(\pi_1\pi_2))}{\langle P,\chi_\alpha\rangle}\neq 0,$$

where $\sigma=(n+1)$-cycle and π_1,π_2 are disjoint cycles of length $(k-t)$ and $(n-k+t)$ respectively.

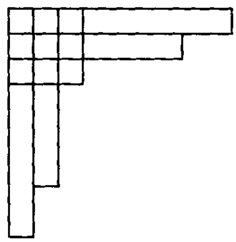
Proof: λ is obviously δ-derived and $B_\lambda=\{(kk{+}1),(tk)\}$. Now, $\epsilon(\lambda,(kk{+}1))=1$ and $\epsilon(\lambda,(tk))=-1$; also $(n{+}1n\cdots 321)(kk{+}1)(tk)$ can be written as the product of disjoint cycles of length $(k-t)$ and $(n-k+t)$ and $(n+1n\cdots321)(tk)(kk+1)$ is an $(n+1)$-cycle. The conclusion now follows from the previous theorem. ∎

To check if

$$\sum_{\alpha \vdash n+1} \frac{\chi_\alpha(1)^2(\chi_\alpha(\sigma) - \chi_\alpha(\pi_1\pi_2))}{\langle P, \chi_\alpha \rangle} \neq 0$$

one has to invoke the Murnagham–Nakayama formula ([5, Theorem 2.4.7]) and the hook formula ([5, Theorem 2.3.21]). It is easy to see that if $\alpha \vdash n+1$ and the Young diagram of α is not included in the shape

$$(k_1, k_2, 3, 2^{k_3}, 1^{k_4}) \quad =$$



then $\chi_\alpha(\sigma) = \chi_\alpha(\pi_1\pi_2) = 0$.

We shall carry out the computation of the sum in the previous corollary in the special case when either $t = k-1$ or $t = 1$ and $k = n$. We have

Corollary 2: *Suppose that either* $\lambda = (1, 3, 5, \ldots, 2n-3, 2n-2, 1) \models n^2$ *or*

$$\lambda = (1, 3, 5, \ldots, 2(k-1)-1, 2, 2(k-1)-1, 2(k+1)-1, \ldots, 2n-1) \models n^2,$$

i.e., λ *is obtained from* δ *by splitting one of its parts into two parts one of length 2. Then* $F^{\lambda,\lambda}(x, y)$ *is a central polynomial for* $M_n(K)$.

Proof: In these cases, recalling that (see [3, p. 105]) if σ is a $(n+1)$-cycle,

$$\sum_{\alpha \vdash n+1} \frac{\chi_\alpha(1)^2\chi_\alpha(\sigma)}{\langle P, \chi_\alpha \rangle} = (-1)^n \frac{n+1}{2n+1},$$

we get that $F^{\lambda,\lambda}(x, y)$ is a central polynomial provided

$$(-1)^n \frac{n+1}{2n+1} - \sum_{\alpha \vdash n+1} \frac{\chi_\alpha(1)^2\chi_\alpha(\pi)}{\langle P, \chi_\alpha \rangle} \neq 0$$

where π is an $(n-1)$-cycle.

Now, $\chi_\alpha(\pi)$ can be computed by using [5, 2.4.7] and [5, 2.3.17]: consider α as a Young diagram on $n+1$ boxes; if, by erasing two boxes from the rim of α, the remaining diagram is not a hook (i.e., of the form $(i, n-i+1)$), then $\chi_\alpha(\pi) = 0$. Also a direct computation shows that for $2 < i < n-1$, $\chi_{(i,2,1^{n-i-1})}(\pi) = 0$ and for $2 < i < n$, $\chi_{(i,1^{n-i+1})}(\pi) = 0$.

Thus the only S_{n+1}-characters giving a nonzero contribution to the above sum are those corresponding to the following partitions:

$(n+1)$	
$(n,1)$	$n \geq 2$
$(n-1,2)$	$n \geq 4$
$(i,3,1^{n-i-2})$	$i \geq 3, n \geq 5$
$(i,2^2,1^{n-i-3})$	$i \geq 2, n \geq 5$
$(2^2,1^{n-3})$	$n \geq 4$
$(2,1^{n-1})$	$n \geq 2$
(1^{n+1})	

We then compute the following tableau:

	(1)	π	$\langle P, - \rangle$
$\chi_{(n+1)}$	1	1	$\frac{(2n+1)!}{n!(n+1)!}$
$\chi_{(n,1)}$	n	1	$\frac{n(2n)!}{(n+1)!(n-1)!}$
$\chi_{(n-1,2)}$	$\frac{(n+1)(n-2)}{2}$	-1	$\frac{(n+1)(2n-1)!}{2(n-1)n!(n-3)!}$
$\chi_{(i,3,1^{n-i-2})}$	$\frac{(n+1)!(i-2)}{2(n-1)(n-i+1)(n-i-2)!i!}$	$(-1)^{n-i+1}$	$\frac{(n+1)(n+2)(i-2)(n+i)!}{2(n-1)(n-i+1)(i+1)!(n-i-2)!i!}$
$\chi_{(i,2^2,1^{n-i-3})}$	$\frac{(n+1)!(n-i-2)}{2(n-1)(i+1)(n-i)!(i-2)!}$	$(-1)^{n-i+1}$	$\frac{n(n+1)(n-i-2)(n+i)!}{2(n-1)(i+1)(i-2)!(i+1)!(n-i)!}$
$\chi_{(2^2,1^{n-3})}$	$\frac{(n+1)(n-2)}{2}$	$(-1)^{n+1}$	$\frac{(n+1)^2(n^2-4)}{4}$
$\chi_{(2,1^{n-1})}$	n	$(-1)^n$	$n(n+2)$
$\chi_{(1^{n+1})}$	1	$(-1)^n$	1

where as α runs over all the above partitions of $n+1$, the columns of the tableau represent the values of $\chi_\alpha(1), \chi_\alpha(\pi)$ and $< P, \chi_\alpha >$ respectively.

Summing over all partitions of $n+1$ we then obtain a nonzero sum for $n \leq 4$

and, for $n \geq 5$, we get

$$(-1)^n \frac{n+1}{2n+1} - \sum_{\alpha \vdash n+1} \frac{\chi_\alpha(1)^2 \chi_\alpha(\pi)}{\langle P, \chi_\alpha \rangle}$$
$$= (-1)^{n+1} \frac{n^2+6n+2}{(2n+1)(n+2)} + \frac{(n+1)!n!(2n^2-5n-4)}{(2n+1)!}$$
$$- \frac{(n+1)!(n-2)!}{2} \times$$
$$\sum_{j=0}^{n-5} \frac{(-1)^j (n-j-4)}{j!(2n-j-3)!(j+3)} \left(\frac{n-j-3}{j+2} - \frac{n(n-j-1)}{(n+2)(2n-j-2)} \right).$$

By using Zeilberger's identity_prover.maple which is an implementation of the method described in [10], it can be shown that the above function is equal to

$$\frac{(-1)^{n+1} 3(n+1)(n^2-2n+2)}{(n+2)(2n-3)(2n-1)};$$

hence it is nonzero. ∎

ACKNOWLEDGEMENT: We would like to thank D. Zeilberger for his help and for making his tools available.

References

[1] V. Drensky, *New central polynomials for the matrix algebras*, Israel Journal of Mathematics **92** (1995), 235–248.

[2] E. Formanek, *Central polynomials for matrix rings*, Journal of Algebra **23** (1972), 129–132.

[3] E. Formanek, *A conjecture of Regev about the Capelli polynomial*, Journal of Algebra **109** (1987), 93–114.

[4] P. Halpin, *Central and weak identities for matrices*, Communications in Algebra **11** (1983), 2237–2248.

[5] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Addison-Wesley, Reading, Mass., 1981.

[6] C. Procesi, *The invariant theory of $n \times n$ matrices*, Advances in Mathematics **19** (1976), 306–381.

[7] Yu. P. Razmyslov, *On a problem of Kaplansky*, Izv. Akad. Nauk SSSR, Ser. Mat. **37** (1973), 483–501 (Russian); Math. USSR Izv. **7** (1973), 479–496 (Translation).

[8] Yu. P. Razmyslov, *Trace identities of full matrix algebras over a field of characteristic zero*, Izv. Akad. Nauk SSSR, Ser. Mat. **38** (1974), 337–360 (Russian); Math. USSR Izv. **8** (1974), 727–760 (Translation).

[9] A. Regev, *The polynomial identities of matrices in characteristic zero*, Communications in Algebra **8** (1980), 1417–1467.

[10] D. Zeilberger, *The method of creative telescoping*, Journal of Symbolic Computation **11** (1991), 195–204.